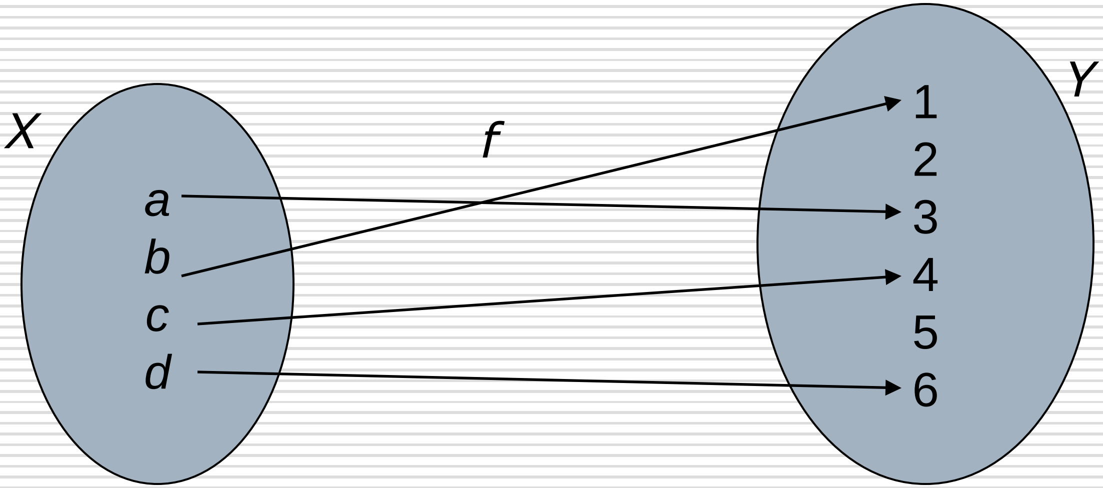
Lecture 2

Learning Objectives:

- To describe functions
- To define a “one-to-one” and “onto” function
- To apply and distinguish special kinds of functions (i.e. absolute value, floor and ceiling, logarithmic, exponential and polynomial)

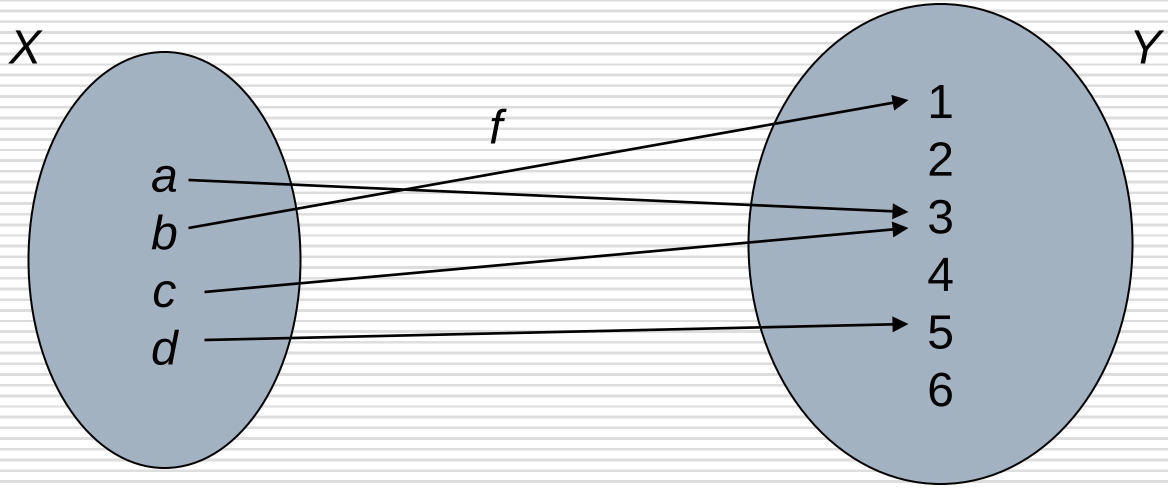
Functions

- Let X and Y be sets. A function from X to Y is a relation that associates an element of Y to every element of X.
- f : $X \rightarrow Y$
  - f is a function from X to Y.



Notice that every element of X has exactly one arrow leaving it.

Functions

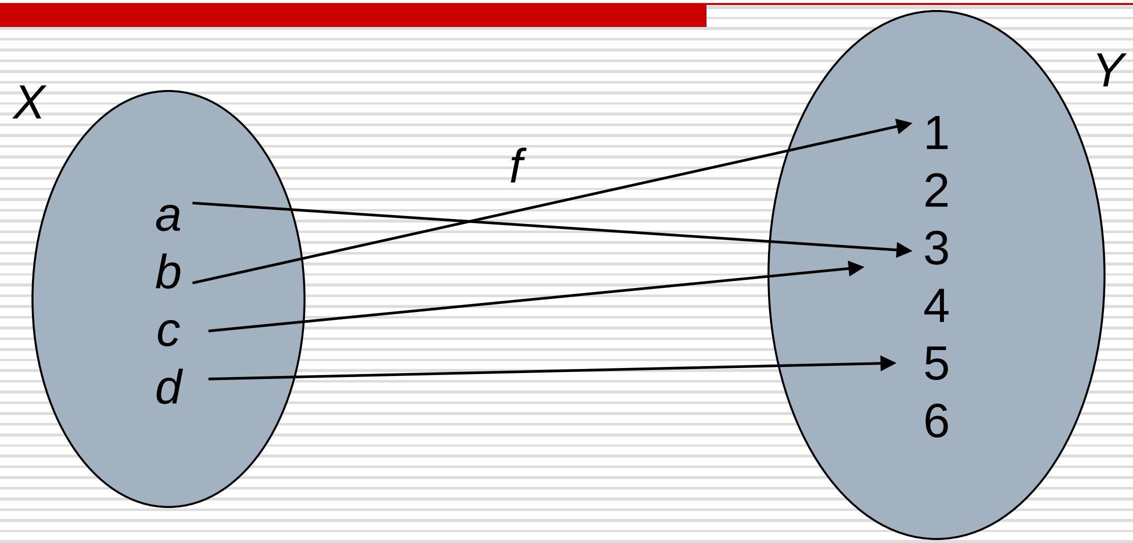


The set X is called the domain of the function f. The set Y is called the codomain of f.

Each arrow takes an element of X to its image in Y. This image is denoted by $f(x)$.

$f(a) = 3$,
$f(b) = 1$,
$f(c) = 3$, and
$f(d) = 5$

Functions



The subset of *Y* consisting of all the images of the elements in *X* is called the range of *f*. It's denoted by f(*X*).

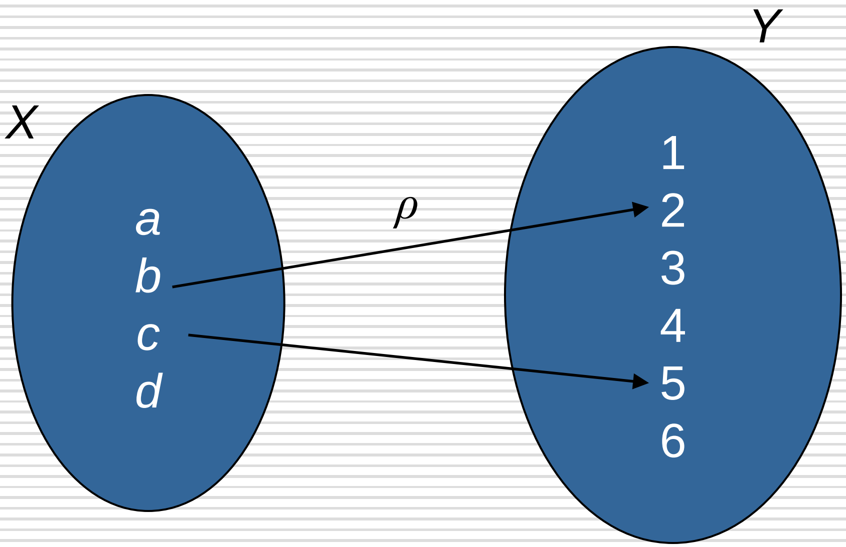
$$f(X) = \{f(x) : x \in X\}$$

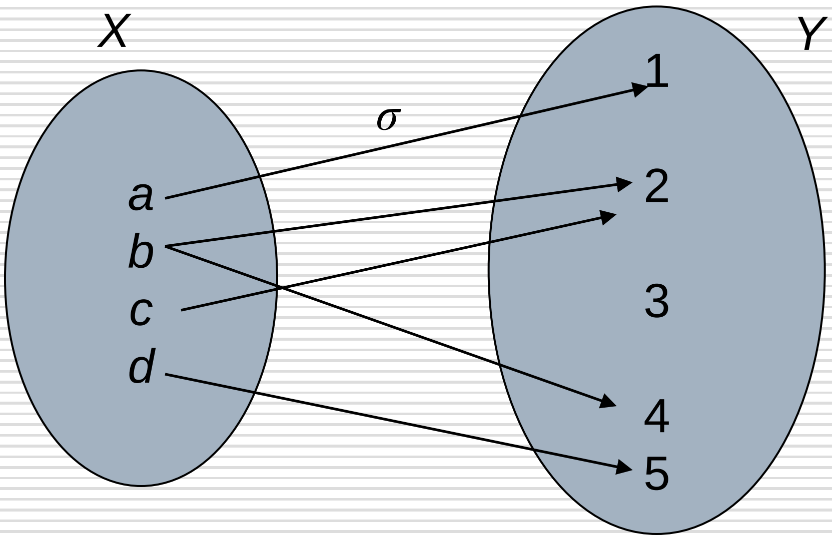
$$f(X) = \{f(a), f(b), f(c), f(d)\} = \{3, 1, 3, 5\} = \{1, 3, 5\}$$

NOTE

Some texts use "range" to mean "codomain", and "image" instead of "range".

Relations that are not functions





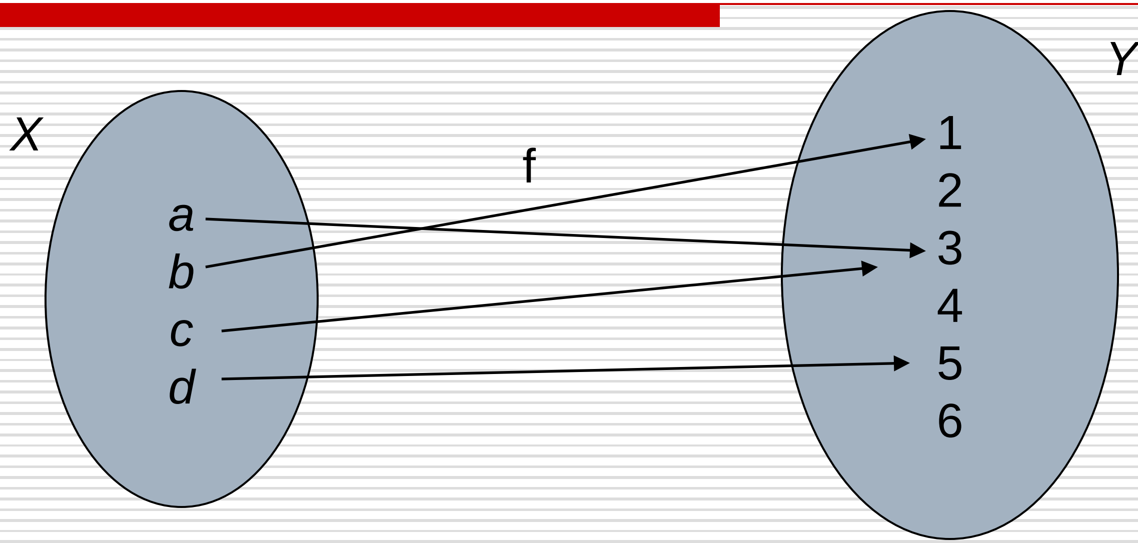
Here ρ is not a function because some elements from X have no image.

Here σ is not a function because some elements have more than one image.

Functions

- Note that functions is a subset of relations.
- A function can be regarded as a relation where every element of the first set occurs exactly once as the first coordinate of an ordered pair.
- The set $\{(x, f(x)): x \in X\}$ completely characterizes the function f (provided that we know the codomain Y)
  - For that reason, sometimes we identify the two concepts and write $f = \{(x, f(x)): x \in X\}$

Functions



$$f(X) = \{f(a), f(b), f(c), f(d)\} = \{3, 1, 3, 5\} = \{1, 3, 5\}$$

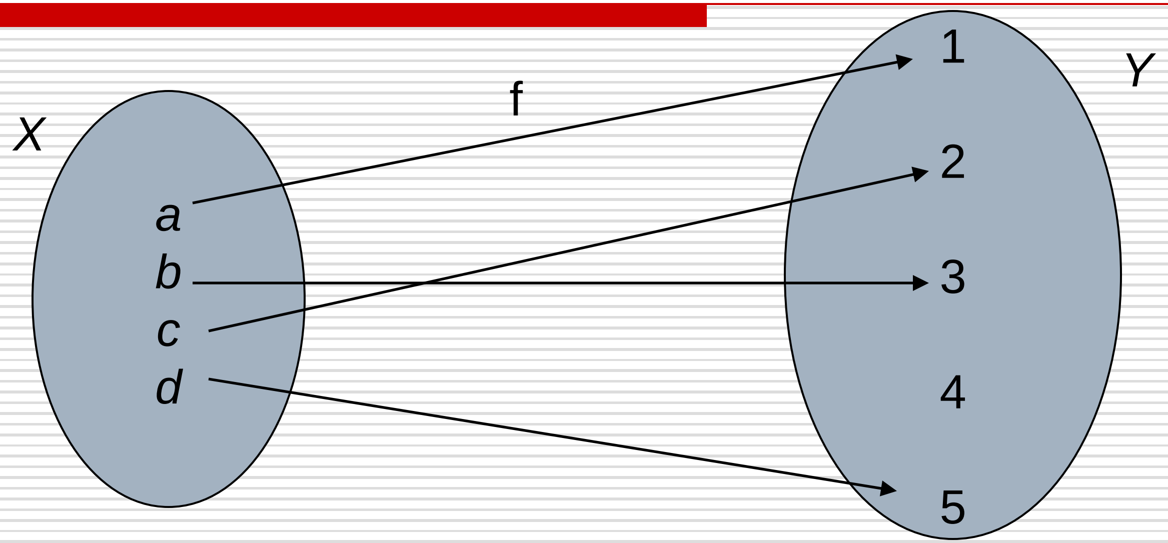
$$f = \{(a, 3), (b, 1), (c, 3), (d, 5)\}$$

Special Kinds of Functions

Functions may be:

- One-to-one
- Onto
- Both one-to-one and onto
- Neither one-to-one nor onto

One-to-one Function

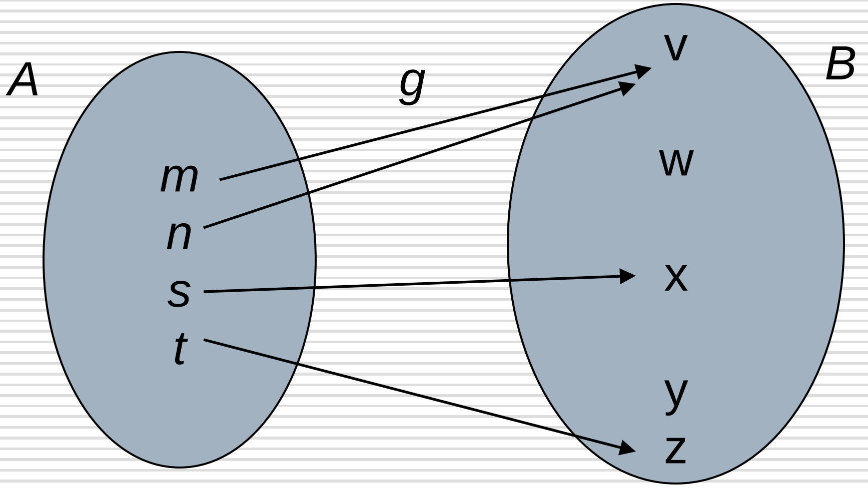


A function f : $X \rightarrow Y$ is one-to-one (or injective, or an injection) if distinct elements of X have distinct images in Y.

For $x_1 = f(x_1)$, $x_2 = f(x_2)$

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

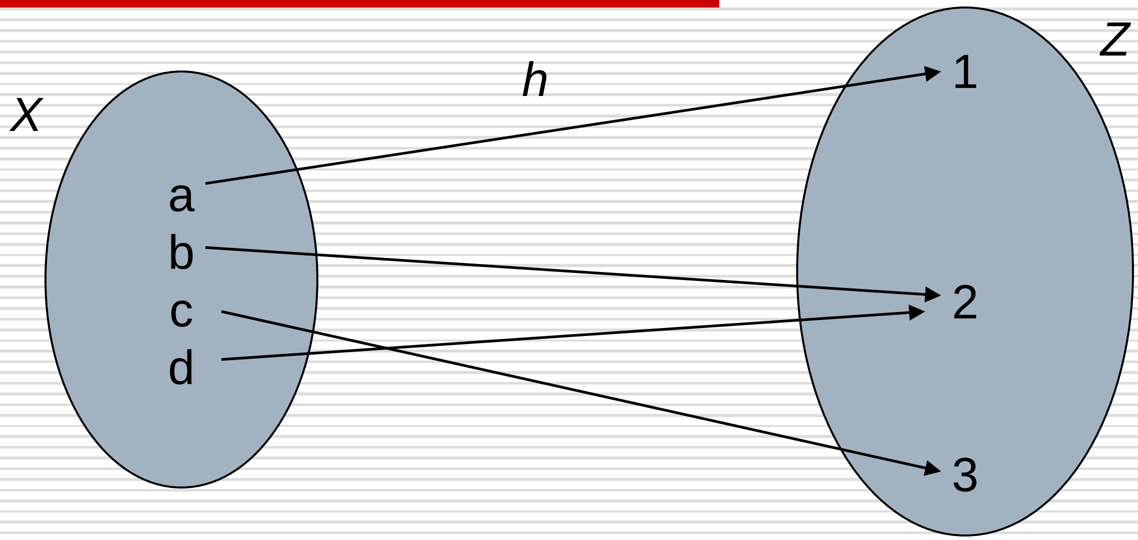
One-to-one Function



Here, *g* is NOT one-to-one.

We see that g(*m*) = g(*n*) = *v*. That is, *m* and *n* have the same image.

Onto Function



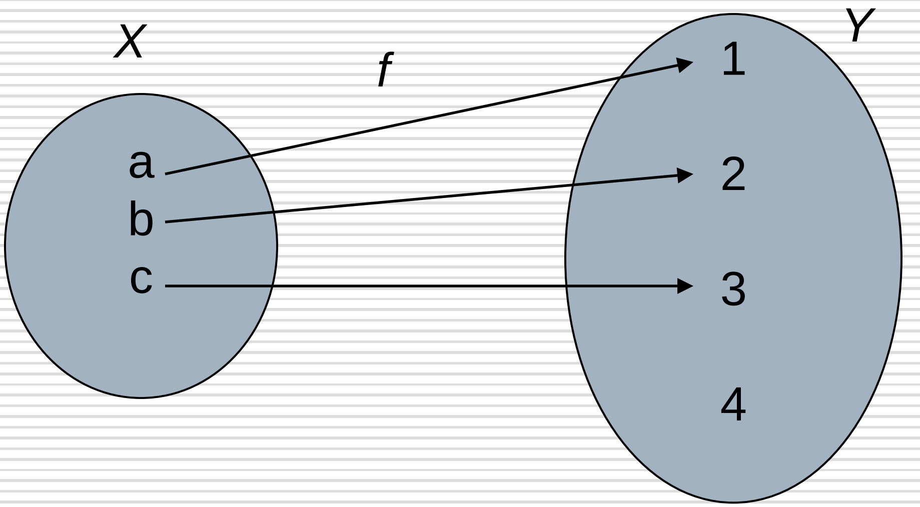
Is *h* one-to-one?

The function *h* is an example of an "onto" function.

Every element of *Z* receives at least one arrow.

Onto Function

- A function f : $X \rightarrow Y$ is onto (or surjective, or a surjection) if $f(X) = Y$
  - The range is the whole of the codomain
  - For every element $y \in Y$, there exists $x \in X$ such that $f(x) = y$
- If $f(x) = y$, so that y is the image of x (under the action of f), we also say that x is a object of y
- So a function is onto if every element of the codomain has a object.



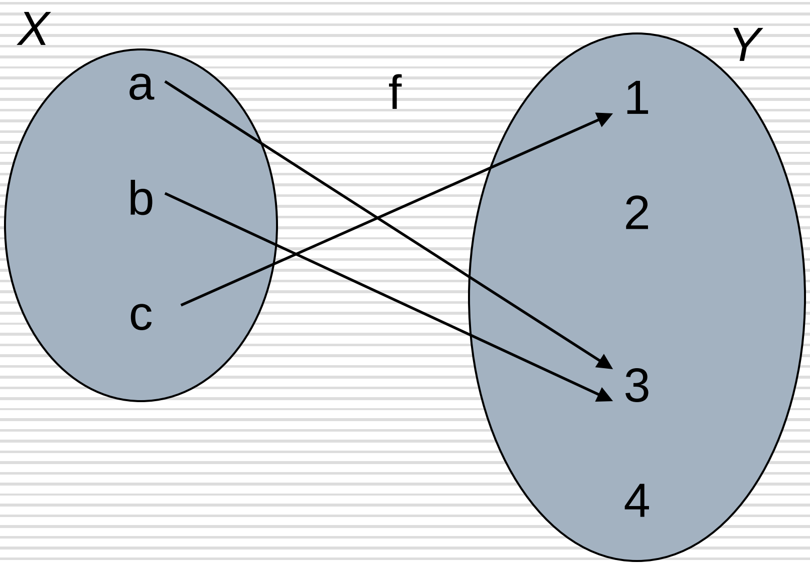
Not onto. Note that 4 is not in the range.

One-to-one and Onto Function

- A function which is both injective (one-to-one) and surjective (onto) is said to be bijective or a bijection
  - A bijection is also called a "one-to-one correspondence"
- For finite sets X and Y, there exists an injection from X to Y if and only if $| X | \leq | Y |$
- There exists a surjection from X onto Y if and only if $| X | \geq | Y |$
- Therefore, there exists a bijection if and only if $| X | = | Y |$

Examples

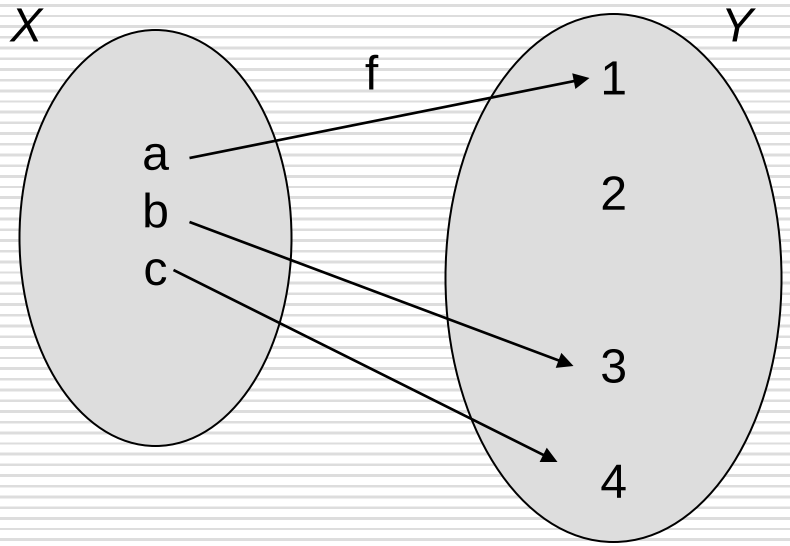
|X|<|Y|



One-to-one? No

Onto? No

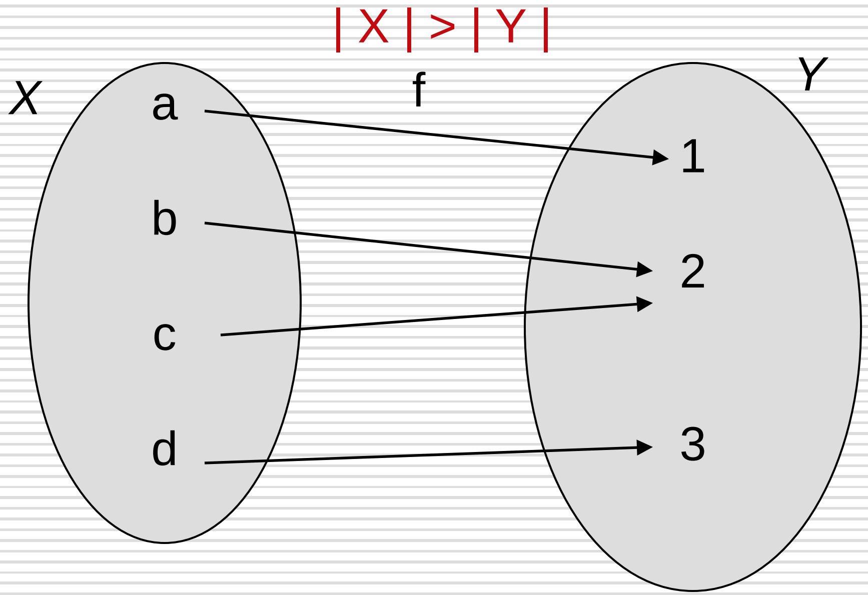
|X|<|Y|

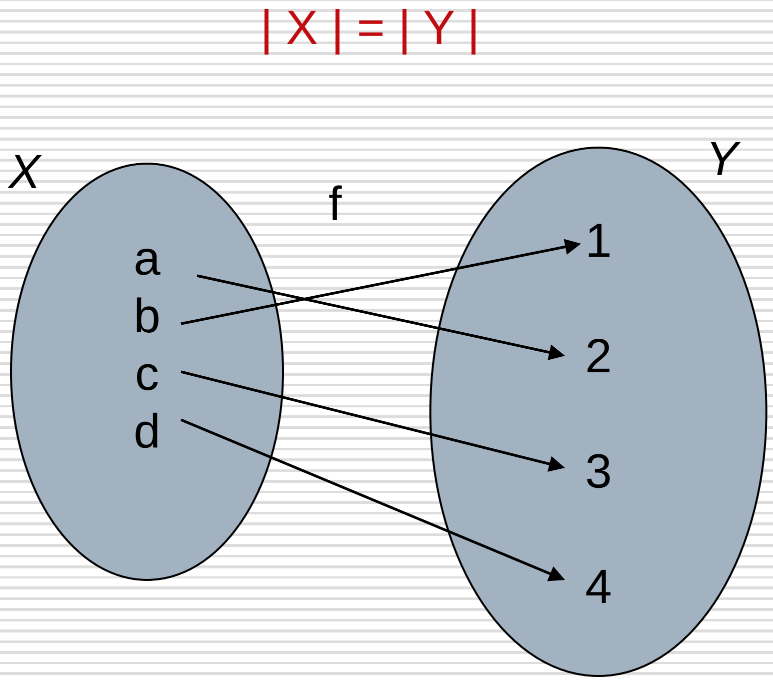


One-to-one? Yes

Onto? No

Examples





One-to-one? No

Onto? Yes

One-to-one? Yes

Onto? Yes

Some Special Functions

- Absolute value
- Floor and ceiling
- Factorial
- Exponential
- Logarithmic
- Polynomial

Absolute Value
Floor and Ceiling

- Absolute value

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

- Example: | -3 | = 3

The absolute value, floor and ceiling functions can be applied to any real number.

So their domain is **R**.

- Floor and ceiling

$$\lfloor x \rfloor = \text{floor of } x$$

$$= \text{first integer} \leq x$$

$$\lfloor 3.79 \rfloor = 3 \qquad \lfloor -3.79 \rfloor = -4$$

$$\lceil x \rceil = \text{ceiling of } x$$

$$= \text{first integer} \geq x$$

$$\lceil 3.79 \rceil = 4 \qquad \lceil -3.79 \rceil = \text{-}3$$

Factorial Function

- This exists for every integer ≥ 0 (the nonnegative integers)
- If $n \geq 1$, $n! = n(n - 1)\ldots 3 \cdot 2 \cdot 1$
- When $n = 0$, $0! = 1$ (special definition)

n	n!
0	1
1	1
2	2
3	6
4	24
5	120
…	…

A partial table

$$n! = n(n - 1)! \qquad \text{for } n \geq 1$$

This is an example of a recurrence relation

Exponential Functions

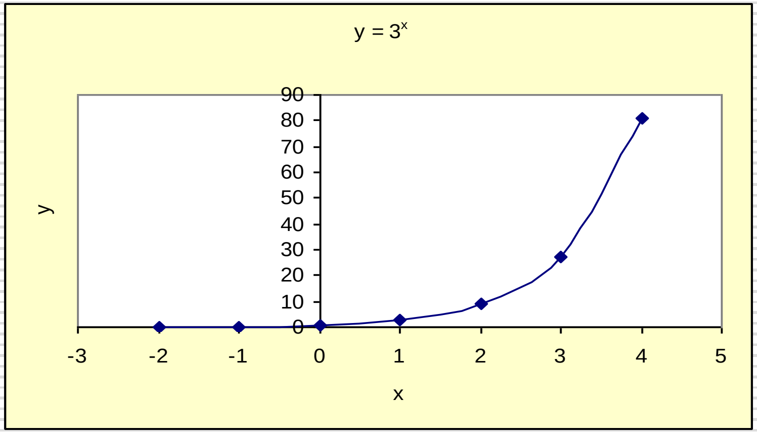
- An exponential function has the form $y = a^x$ for some constant $a > 0, a \neq 1$

- We call a the base and x the exponent.

- Then $y = a^x$ is a power of a

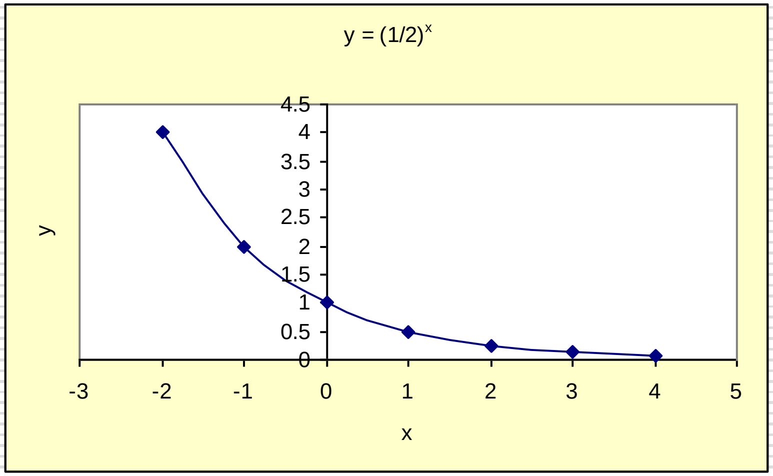
Exponential Functions

- For $a > 1$, an exponential function grows very rapidly
- So algorithms or programs that take exponential time or space can't practically be implemented except for small values of the parameters



As $x \rightarrow \infty$, $y \rightarrow \infty$ (rapidly)
and as $x \rightarrow -\infty$, $y \rightarrow 0$.

- On the other hand, if $0 < a < 1$ we have a function that models radioactive decay.



As $x \rightarrow \infty$, $y \rightarrow 0$
and as $x \rightarrow -\infty$, $y \rightarrow \infty$.

Laws of Exponents

$$a^x a^y = a^{x+y}$$ **Example** $$3^2 \cdot 3^5 = 3^{2+5} = 3^7$$

$$a^x / a^y = a^{x-y}$$ **Example** $$3^2 / 3^5 = 3^{2-5} = 3^{-3}$$

$$a^0 = 1$$

$$a^{-x} = \frac{1}{a^x}$$ **Example** $$3^{-3} = \frac{1}{3^3} = \frac{1}{27}$$

$$(a^x)^y = a^{xy}$$ **Example** $$(3^2)^5 = 3^{2(5)} = 3^{10}$$

Logarithmic Functions

- We write $y = \log_a x$ to mean that $a^y = x$
  - Here, y is the "log of x, to the base a"

$$\boxed{\log_a x = y \Leftrightarrow a^y = x}$$

- Example 1:
  Find the following

  a. $\log_5 25 =$

$$let \log_5 25 = x$$
$$5^x = 25$$
$$5^x = 5^2$$
$$x = 2$$

Logarithmic Functions

b. $\log_3 81 = ?$

$$\text{Let } \log_3 81 = x$$
$$3^x = 81$$
$$3^x = 3^4$$
$$x = 4$$

c. $\log_2 128 = ?$

$$\text{Let } \log_2 128 = x$$
$$2^x = 128$$
$$2^x = 2^7$$
$$x = 7$$

Logarithmic Functions

d. $\log_3 1 = ?$

Let $\log_3 1 = x$

$$3^x = 1$$
$$3^x = 3^0$$
$$x = 0$$

Laws of Logarithms

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a x^n = n \log_a x$$

$$\left.\begin{array}{c} \log_a a = 1 \\ \log_a 1 = 0 \\ \log_a a^{-1} = -1 \end{array}\right\}$$

$$\log_a a^n = n$$

special cases

Example

$$\log_2 \frac{1}{4} = \log_2 4^{-1}$$

$$= -1 \cdot \log_2 4$$

$$= -2$$

$$\log_2 8 = \log_2 (2 \times 4)$$

$$= \log_2 2 + \log_2 4$$

$$= 1 + 2 = 3$$

Useful Bases

- $\log x = \log_{10} x$ (many calculators)
- $\ln x = \log_e x$ ($e \approx 2.71828$)
- $\lg x = \log_2 x$ (some texts)

Polynomial Functions

- A function of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

is called a polynomial function.

- The expression on the right of the "equals" sign is a polynomial.

$$\underbrace{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n}$$

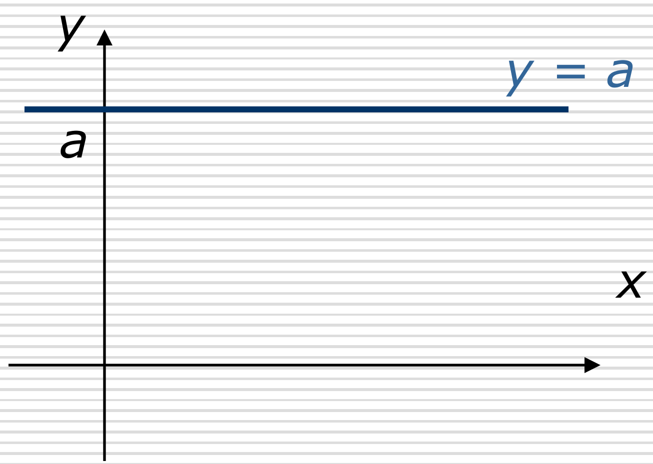
a general polynomial

If $a_n \neq 0$, this has degree n.

Examples of Polynomial Functions $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

- $n = 0, f(x)$ is a Constant Functions

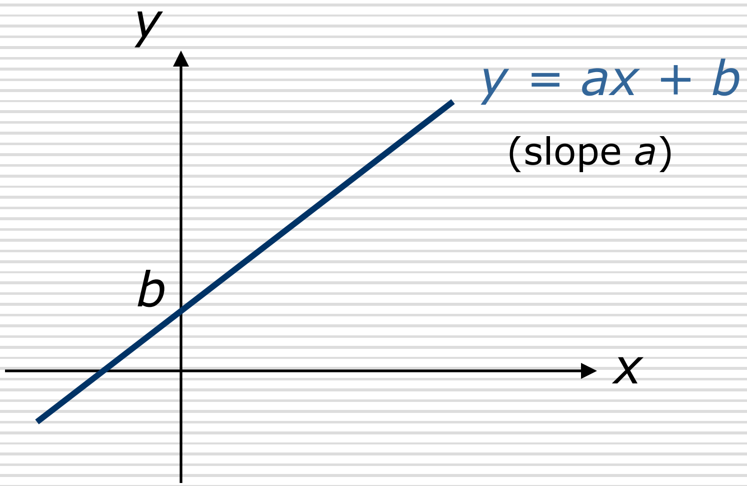
  $y = a$



  - Degree 0 (except *a* = 0)

- $n = 1, f(x)$ is a Linear Functions

  $y = a_0 + a_1 x$

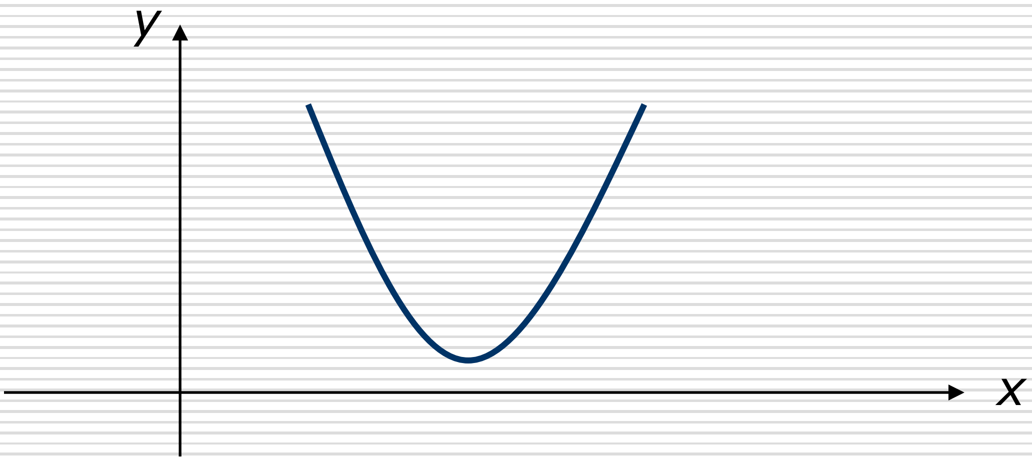


  - Degree 1

Examples of Polynomial Functions $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

- n=2, $f(x)$ is a Quadratic Functions

$$y = a_0 + a_1 x + a_2 x^2$$



- Degree 2

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